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STABILIZATION OF CONTROLLED SYSTEMS BY NONLINEAR CONTROL ACTIONS

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Sufficient conditions are obtained for the asymptotic stability under any initial conditions for a system controlled by nonlinear feedback conforming to the state vector The conditions are stated as explicit analytic relations. The results obtained supplement the investigations in [1-3] and use the results in [4-8].

1. Statement of the problem. Formulation of the results. Let R^n be the n-dimensional arithmetic space of points $x = \mathrm{col}\,\{x_1, \ldots, x_n\}$, $\| a_{sl} \|$ be a matrix, δ_{ij} be the Kronecker symbol, and T be the symbol of transposition. Let the indices ν and μ run from 1 to m, the indices i and j from 1 to n_ν and n_μ, respectively, and the indices l and s from 1 to n and $m \leqslant n$. Let $A = \| A_{\nu\mu} \|$ denote a block matrix with blocks $A_{\nu\mu} = \| a_{ij}^{\nu\mu} \|$, where $a_{ij}^{\nu\mu}$ are elements of a block. We examine a controlled object whose motion is described by the differential equation with constant parameters

$$x^{\cdot} = Ax + Bu, \quad x \in R^n, \quad u \in R^m \tag{1.1}$$
$$t \in [t_0, \infty), \quad x(t_0) = x^\circ$$

where A and B are an $n \times n$-dimensional matrix and an $n \times m$-dimensional matrix, respectively.

Problem. For system (1.1) let

$$u^\circ(x) = M(x)\, x \tag{1.2}$$

be the control law for which the closed system

$$x^{\cdot} = Ax + BM(x)\, x, \quad x(t_0) = x^\circ \tag{1.3}$$

is asymptotically stable under any initial conditions $t_0 \geqslant 0$ and $| x_s^\circ | < N$, $N < \infty$. Find the conditions that the elements of matrix $M(x)$ satisfy in this connection.

To solve the problem posed it is necessary to investigate the existence of the functional

$$J(C, u) = \frac{1}{2} \int_{t_0}^{\infty} [C(x) + u^T u]\, dt \tag{1.4}$$

($C(x)$ is a positive definite function) that the control forces $u^\circ(x)$ minimize and thus optimally stabilize system (1.1). The inverse problem of optimal control [9] arises in this case. In the present paper the results in [5-8] on the inverse problem are extended to that of nonlinear control. The basic tool for solving this problem is the canonic transformation [10] of the original equations of motion of the controlled object.

Let for system (1.1), the pair (A, B) be completely controllable and let matrix B have full rank. Then according to [10] the nondegenerate transformations $y = Tx$ and $v = Gu$ exist for which

$$y^{\cdot} = Fy + Hv, \quad y \in R^n, \quad v \in R^m \tag{1.5}$$

$$t \in [t_0, \infty), \quad y(t_0) = y^{\circ}, \quad t_0 \geqslant 0 \quad |y_s^{\circ}| < N' \tag{1.6}$$

$$F = \| F_{\nu\mu} \|, \quad F_{\nu\mu} = \| f_{ij}^{\nu\mu} \|, \quad H = \| \delta_{l\sigma(\nu)} \|$$

$$f_{ij}^{\nu\mu} = \delta_{\nu\mu}\delta_{ij-1} + \delta_{in(\nu)} a_{\nu\delta(\mu-1)+j}, \quad n(\nu) = n_\nu, \quad n(0) = 0, \quad \sigma(\nu) = n_1 + \ldots + n_\nu, \quad \sigma(0) = 0, \quad \sigma(m) = n$$

Definition. Two performance indices $J(C_1, u)$ and $J(C_2, u)$ are said to be equivalent if they are minimized by the same control $u(x)$.

Lemma. In a performance index $J(C', v)$ let the function $C'(y)$ be positive definite and let $C'(y) \to \alpha y^T y$ as $y \to 0$, where $\alpha \geqslant 0$. Then functional $J(C', v)$ is equivalent to $J(L, v)$, where

$$L(y) = y^T Q(y) y, \quad Q = \| q_{ij}^{\nu\mu} \|$$

$$q_{ij}^{\nu\mu}(y) = d_i^{\nu\nu}(y)\delta_{ij}\delta_{\nu\mu} + {}^1/_2(1 - \delta_{\nu\mu})\{d_i^{\nu\mu}(y)\delta_{1j}[1 + \operatorname{sign}(n_\nu + 1 - i - j)] + d_{n(\nu)+j-1}^{\nu\mu}(y)\delta_{in(\nu)}[1 - \operatorname{sign}(n_\nu + 1 - i - j)]\} \quad (\nu \neq \mu)$$

The Lemma is an extension of the result in [8] to the nonlinear case and states that a functional of a special structure can be selected from the class of equivalent performance indices. This selection significantly decreases the number of parameters in the functional and permits the unique solution of the inverse optimal control problem. In particular, for systems with a one-dimensional control the functional $J(C', v)$ is equivalent to a functional $J(L, v)$ with a diagonal matrix $Q(y)$.

Theorem 1. Let in the control law

$$v^{\circ}(y) = K(y)y, \quad K = [K_1, \ldots, K_m], \quad K_\nu = \| k_{\mu i}^{\nu} \| \tag{1.7}$$

the elements of matrix K be bounded and satisfy the equalities $k_{\nu n(\mu)}^{\mu}(y) \equiv k_{\mu n(\nu)}^{\nu}(y)$, and let the control forces (1.7) minimize the performance index $J(L, v)$. Then the elements of matrix $Q(y)$ are defined in terms of the elements $k_{\mu i}^{\nu}(y)$ thus:

$$d_1^{\nu\nu}(y) = \xi_{11}^{\nu\nu}(y), \quad d_i^{\nu\nu}(y) = \xi_{ii}^{\nu\nu}(y) + 2\sum_{\lambda=0}^{\varkappa}(-1)^{\lambda+1}\xi_{i-\lambda-1\; i+\lambda+1}^{\nu\nu}(y) + (-1)^{n(\nu)-i}[1 - \operatorname{sign}(n_\nu + 1 - 2i)]\, k_{\nu 2i-n(\nu)-1}^{\nu}(y) \tag{1.8}$$

$$(i > 1, \; \mu = \nu, \; i = j)$$

$$d_i^{\nu\mu}(y) = \sum_{\lambda=0}^{\varkappa}(-1)^{\lambda}\xi_{i-\lambda\; i+\lambda}^{\nu\mu}(y) + \frac{1}{2}(-1)^{n(\mu)}[1 - \operatorname{sign}(n_\nu - i)] \times$$

$$\times\ k^{\nu}_{\mu\ i-n(\mu)}(y) \quad (j=0), \qquad d^{\nu\mu}_{n(\nu)+j-1}(y) =$$

$$\sum_{\lambda=0}^{n(\mu)-j} (-1)^{\lambda}\xi^{\nu\mu}_{n(\nu)-\lambda j+\lambda}(y) + (-1)^{n(\mu)-j+1}k^{\nu}_{\mu n(\nu)-n(\mu)-1+j}(y) - k^{\mu}_{\nu j-1}(y)$$

$$(\nu \neq \mu)$$

$$\varkappa = \min(i-1, n_\mu - j - 1), \quad \|\xi^{\nu\mu}_{ij}(y)\| = (\Lambda - H^T I)^T K(y) + K^T(y)(\Lambda - H^T I) + K^T(y) K(y), \quad \Lambda = \|a_{\nu s}\|, \quad I = \|\delta_{sl-1}\|$$

Another result concerning the inverse optimal control problem for a linear stationary system with nonlinear control was obtained in [11].

Theorem 2. Let the hypotheses of Theorem 1 be fulfilled and $Q(K, y)$ be a matrix whose elements are calculated by (1.8). If the matrix $Q(K, y) + K^T(y) K(y)$ is positive definite, the control forces (1.7) stabilize system (1.5) under any initial conditions (1.6).

The latter assertion is a sufficient condition of asymptotic stability for the system $y^{\cdot} = Fy + HK(y)\,y$ under any initial conditions (1.6).

2. Proof of the results. Proof of the Lemma. Let the equalities

$$B^T \Delta P(x) = 0, \quad A^T \Delta P(x) + \Delta P(x) A + \Delta S(x) = 0 \qquad (2.1)$$

be valid for the symmetric $n \times n$ -matrices $\Delta P(x)$ and $\Delta S(x)$. Then the functional $J(C + x^T \Delta S x,\ u)$ is equivalent to functional (1.4). Indeed, the minimum of the first is achieved for

$$u'(x) = -B^T\left[\frac{\partial V'}{\partial x}\right]^T, \quad 2\frac{\partial V'}{\partial x}Ax + C(x) + x^T \Delta S(x)\,x - \frac{\partial V'}{\partial x}BB^T\left[\frac{\partial V'}{\partial x}\right]^T = 0$$

where $V'(x)$ is the Bellman function in the case of steady-state motion. With due regard to (2.1) the last relation can be rewritten as

$$u'(x) = -B^T P'(x)x, \quad x^T\{A^T[P'(x) - \Delta P(x)] + [P'(x) - \Delta P(x)]A + [P'(x) - \Delta P(x)]BB^T[P'(x) - \Delta P(x)]\} + C(x) = 0 \qquad (2.2)$$

by setting $\partial V'/\partial x = x^T P'(x)$, where $P'(x)$ is a positive definite symmetric $n \times n$ -matrix. Analogous relations can be obtained in the minimization of functional (1.4). By comparing them with (2.2) we obtain $P'(x) - \Delta P(x) = P(x)$, whence, with due regard to (2.1), follows $u'(x) = -B^T[P(x) + \Delta P(x)]x = u(x)$.

We need to show that $J(L, v)$ exists among the functionals equivalent to $J(C', v)$. Let $C'(y) = y^T S(y)\,y$; we set $Q(y) = S(y) + \Delta S'(y)$, where the elements of matrix $\Delta S'(y)$ are found from the equalities $\Delta p^{\nu\mu}_{n(\nu)j}(y) = \Delta p^{\nu\mu}_{in(\mu)}(y) = 0$ and $\Delta p^{\nu\mu}_{i-1j}(y) + \Delta p^{\nu\mu}_{ij-1}(y) + \Delta s^{\nu\mu}_{ij}(y) = 0$, which are the analogs equalities (2.1) taken element by element. In the last equalities there are $\Sigma_{\nu<\mu}(n_\nu - 1)(n_\mu - 1) + \Sigma_\nu (n_\nu - 1)\, n_\nu / 2$ arbitrary elements $\Delta p_{ij}{}^{\nu\mu}(y)$. When

$$\Delta p^{\nu\mu}_{ij}(y) = \sum_{\lambda=0}^{\varkappa} (-1)^{\lambda} s^{\nu\mu}_{i-\lambda\ j+\lambda+1}(y)$$

matrix $Q(y)$ has the form given in the Lemma.

Proof of Theorem 1. The optimal control

$$v(y) = -H^T P(y)\, y, \quad y^T \{F^T P(y) + P(y) F + Q(y) - P(y) HH^T P(y)\}\, y = 0 \tag{2.3}$$

is obtained for system (1.5) and performance index $J(L, v)$. Let matrix $Q(y)$ be calculated by formulas (1.8) for a specified $v^\circ(y)$. Then the matrix $P = \| p_{ij}^{\nu\mu} \|$ calculated by the formulas

$$p_{ij}^{\nu\mu}(y) = \sum_{\lambda=0}^{\varkappa} (-1)^\lambda \xi_{i-\lambda\, j+\lambda+1}(y) + (-1)^{n(\mu)-j} \times [1 - \operatorname{sign}(n_\mu - i - j)]\, k^{\nu}_{\mu\, i+j-n(\mu)}(y) \tag{2.4}$$

$$p^{\nu\mu}_{n(\nu)j}(y) = -k^{\mu}_{\nu j}(y), \quad p^{\nu\mu}_{in(\mu)}(y) = -k^{\nu}_{\mu i}(y)$$

satisfies (2.3) and $v(y) = v^\circ(y)$. The theorem is proved.

Proof of Theorem 2. Proof of the last statement reduces to verifying the fulfilment of the hypotheses of Theorem IV in [4]. From Theorem 1 it follows that when $L(y) = y^T Q(K, y)\, y$ the equality $\partial W / \partial y\, [Fy + Hv^\circ(y)] + {}^1/_2 L(y) + {}^1/_2 [v^\circ(y)]^T v^\circ(y) = 0$ is fulfilled, where $\partial W / \partial y = y^T P(y)$, the elements $p_{ij}^{\nu\mu}(y)$ are calculated by (2.4) and the inequality $\partial W / \partial y\, (Fy + Hv) + {}^1/_2 L(y) + {}^1/_2 v^T v \geqslant 0$ is valid for any number v_ν.
When the theorem's hypotheses are fulfilled the function $y^T [Q(K, y) + K^T(y) K(y)]\, y$ is positive definite and, consequently, so is the function $W(y) = \min J(L, v) = J(y^T Q(K, y)\, y,\ v^\circ(y))$. Thus, all the hypotheses of Theorem IV in [4] are fulfilled; consequently, the control law (1.7) optimally stabilizes system (1.5) relative to performance index $J(L, v)$ under any initial conditions (1.6). The theorem is proved.

3. Examples. 1°. We consider a system with a one-dimensional control

$$\begin{gathered} y^{\cdot} = Fy + hv, \quad y \in R^n, \quad v \in R, \quad t \in [t_0, \infty) \\ y(t_0) = y^0, \ v^0(y) = f(\sigma), \ \sigma = \alpha^T y \\ f(\sigma)/\sigma < \Gamma \ (0 < \Gamma < \infty), \ F = \| f_{sl} \| \\ f_{sl} = \delta_{sl-1} + \delta_{sn} a_l, \quad \alpha = \operatorname{col}\{\alpha_1, \ldots, \alpha_n\}, \quad h = \| \delta_{sn} \| \end{gathered} \tag{3.1}$$

It is required to find conditions on the function $\varphi(\sigma) = f(\sigma)/\sigma$ and the parameters α_s for which system (3.1) is asymptotically stable under any initial conditions (1.6). To obtain the results we reduce the control $v^0(y)$ to the form $v^0(y) = k^T(y) y$, where $k^T(y) = \varphi(\sigma)\alpha^T$. We noted earlier that the matrix $Q(y)$ in functional $J(L, v)$ can be chosen to be diagonal, $Q = \operatorname{diag}\{d_1, \ldots, d_n\}$. Then from (1.8) follows

$$d_s(y) = \varphi(\sigma)\, [\beta_s + \gamma_s \varphi(\sigma)]$$

$$\beta_s = 2\sum_{\lambda=0}^{\varkappa}(a_{s-\lambda}\alpha_{s+\lambda} + a_{s+\lambda}\alpha_{s-\lambda}) + (-1)^{n-s}[1 - \operatorname{sign}(n+1-2s)] \times$$
$$\alpha_{2s-n-1}, \quad \gamma_1 = \alpha_1^2$$
$$\gamma_s = \alpha_s^2 + 2\sum_{\lambda=0}^{\varkappa}(-1)^{\lambda+1}\alpha_{s-\lambda-1}\alpha_{s+\lambda+1} \quad (s > 1)$$

From Theorem 2 it follows that system (3.1) is asymptotically stable under any initial conditions (1.6) if the inequalities $d_1(y) > 0, \ldots, d_n(y) > 0$ are fulfilled. From these inequalities follow the inequalities $\varphi(\sigma) > 0$, $\varphi(\sigma) > -\beta_s/\gamma_s$ or $\varphi(\sigma) > r$, where $r = \max(0, -\beta_1/\gamma_1, \ldots, -\beta_n/\gamma_n)$. Hence, with due regard to the constraints, we obtain

$$r < f(\sigma)/\sigma < \Gamma, \quad 0 < r < \Gamma \tag{3.2}$$

Thus, conditions (3.2) yield the solution of the problem posed.

Note. In the general case it is necessary to examine all the inequalities under which the conditions $d_s(y) > 0$ can be fulfilled. However, in the example analyzed we can restrict ourselves to just the inequalities determined, first of all, by the inequality $\varphi(\sigma) > 0$ or $f(\sigma)/\sigma > 0$, since the opposite inequalities $\varphi(\sigma) < 0$ and $\varphi(\sigma) < -\beta_s/\gamma_s$ can be reduced to the previous ones by the following changes of signs: $\alpha^* = -\alpha$, $\sigma^* = -\sigma$, $\beta_s^* = -\beta_s$, $\gamma_s^* = \gamma_s$, and $\varphi^*(\sigma^*) = -f(\sigma)/\sigma$, i.e., $\varphi^*(\sigma^*) > 0$ and $\varphi^*(\sigma^*) > -\beta_s^*/\gamma_s^*$.

2°. For Bulgakov's first problem [2] we have the equations

$$T^2\psi^{\cdot\cdot} + U\psi^{\cdot} + \mu = 0, \quad \mu^{\cdot} = f(\sigma) \tag{3.3}$$
$$\sigma = a\psi + E\psi^{\cdot} - l^{-1}\mu, \quad \psi(t_0) = \psi_0$$
$$\psi^{\cdot}(t_0) = \psi_0^{\cdot}, \quad \mu(t_0) = \mu_0, \quad t_0 \geqslant 0 \tag{3.4}$$
$$\{|\psi_0|, \ |\psi_0^{\cdot}|, \ |\mu_0|\} < N$$

After reducing (3.3) to form (3.1) we obtain

$$\alpha = \operatorname{col}\left\{-\frac{a}{T^2}, \ -\frac{El+U}{lT^2}, \ -\frac{1}{l}\right\}$$

To simplify the final result we take advantage of the obvious property: the optimal control problem for system (3.1) relative to the performance index $J(C', v + a^Ty)$, where $a = \operatorname{col}\{a_1, \ldots, a_n\}$, is equivalent to the optimal control problem for the system $y^{\cdot} = (F - ha^T)y + hy$ relative to the performance index $J(C', v)$. Then for Bulgakov's problem we obtain

$$J(L, v) = \frac{1}{2}\int_0^\infty [d_1(y)y_1^2 + d_2(y)y_2^2 + d_3(y)y_3^2 + v^2]\,dt$$
$$d_1(y) = \alpha_1^2\varphi^2(\sigma), \quad d_2(y) = (\alpha_2^2 - 2\alpha_1\alpha_3)\varphi^2(\sigma), \quad d_3(y) =$$
$$(\alpha_3^2\varphi(\sigma) + 2\alpha_2)\varphi(\sigma)$$

System (3.3) is asymptotically stable under any initial conditions (3.4) if the inequalities

$d_1(y) > 0, d_2(y) > 0, d_3(y) > 0$ or $\alpha_2^2 - 2\alpha_1\alpha_3 > 0$ and $\varphi(\sigma) > \max(0, -2\alpha_2/\alpha_3^2)$ are fulfilled. Finally, we obtain the conditions

$$\left(\frac{El+U}{\sqrt{alT}}\right)^2 > 2, \quad \max\left(0, \frac{2l(lE+U)}{T^2}\right) < \frac{f(\sigma)}{\sigma} < \Gamma$$

A comparison of the last result with others for the same problem showed that the resulting stability boundaries are wider than those in [2]. This result was simple to derive since the basic inequalities do not contain arbitrary constants and complex quantities. The latter facilitates the derivation of numerical results.

REFERENCES

1. Lur'e, A.I., Certain Nonlinear Problems in Automatic Control Theory. Moscow. Gostekhizdat, 1951.
2. Letov, A.M., Stability of Nonlinear Controlled Systems. Moscow, Fizmatgiz, 1962.
3. Aizerman, M.A., Theory of Automatic Control of Motors. Moscow, "Nauka", 1966.
4. Krasovskii, N.N., Problems of the stabilization of controlled motions. In: Theory of Stability of Motion. Supplement 4. Moscow, "Nauka", 1966. (see also English translation, Stability of Motion, Stanford Univ. Press, Stanford, Cal. 1963).
5. Das, P.K., On the direct and inverse optimization problems for quadratic functionals in linear autonomous and controlled systems. Autom. Telemekh., No.9, 1966.
6. Aleksandrov, A.G. and Podchukaev, V.A., On the inverse problem of analytic design of regulators for variable-parameter systems. Izv. Vuzov, Priborostroenie, Moscow, No.3, 1975.
7. Gal'perin, E.A., Spectral equivalence of functionals and solvability of optimal stabilization problems. Izv. Akad. Nauk SSSR, Tekhn. Kibernet., No. 4, 1967.
8. Jamson, A. and Kreindler, E., Inverse problem of linear optimal control. SIAM J. Control, Vol. 11, No.1, 1975.
9. Letov, A.M., Selection of the optimizing functional in the analytic design problem. In: Optimal Systems. Statistical Methods. Moscow, "Nauka", 1967.
10. Brunovský, P., On the stabilization of linear systems under a specific class of constantly acting perturbations. Differents. Uravn., Vol. 2, No. 6, 1966.
11. Yokotama, R. and Kinnen, E., The inverse problem of the optimal regulator. IEEE Trans. Auto. Control, Vol. AC-17, No.4, 1972.

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